Scholarly Research Journal for Interdisciplinary Studies
Online ISSN 2278-8808, SJIF 2021 = 7.380,
http://www.srjis.com/issues_data?issueId=213
PEER REVIEWED & REFEREED JOURNAL, MAY-JUNE 2023, VOL- 11/77
https://doi.org/10.5281/zenodo.8214247

STOCHASTIC CONTROLLABILITY OF NONLINEAR SYSTEMS WITH TIME VARIABLE DELAY IN CONTROL

R. B. Sonawane

Department of Mathematics and Statistics, Ness Wadia College of Commerce, Pune (M.S.), India – 411001. sonawaneramdas@gmail.com

Paper Received On: 21 JUNE 2023

Peer Reviewed On: 30 JUNE 2023

Published On: 01 JULY 2023

Abstract

We consider the stochastic controllability problem for nonlinear infinite dimensional systems with time variable delay in controls. Controllability problem for infinite dimensional systems has been studied by numerous researchers. Most of the researchers have considered the linear or semilinear stochastic systems with fixed time delay in control. Some of the researchers have studied controllability of deterministic semilinear systems with fixed delay in control. In this article we have studied the controllability of stochastic nonlinear infinite dimensional systems with time variable delay in control. We have illustreted the results obtained in this article with some examples.

Key words: *Controllability; Nonlinear systems; Time variable delay; Stochastic systems.*

I. Introduction

Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{U}$ be separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. $\mathbb{E}(\cdot)$ denotes the expectation with respect to measure $\mathbb{P}$. Consider the nonlinear system with variable delay in control as follows:

$$\begin{aligned} dx(t) = \;& (Ax(t) + B_1 u(t) + B_2 u(v(t)) + f(t, x(t), u(t))dt \\ & + g(t, x(t), u(t))dw(t), \qquad 0 \leq t \leq T, \end{aligned} \tag{1}$$

with initial conditions

$$x_0 = x_0 \in \mathcal{L}_2(\Omega, \mathcal{F}_0, \mathcal{H}) \qquad \text{and} \qquad u(t) = 0, \qquad t \in [v(0), 0]. \tag{2}$$

$A: D(A) \subset \mathcal{H} \to \mathcal{H}$ is a closed linear operator generating strongly continuous semigroup $S(t)$, $B_1,\ B_2 \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ are bounded linear operators and $\{w(t): t > 0\}$ is a given $\mathcal{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q > 0$. Let $\mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ be the space of all Q -Hilbert-Schmidt operators $g: [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ with norm $\|\cdot\|$ defined as $\| g \|_Q = Tr(gQg^T)$ and $f: [0, T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{H}$. $\mathcal{L}_2^{\mathcal{F}}([0, T] \times \mathcal{H}, \mathcal{H})$ is the space of all $\mathcal{F}_t$-adapted, $\mathcal{H}$-valued measurable square integrable processes with the norm $\|\cdot\|_{\mathcal{H}}$. $v(t) = t - h_1(t)$ is continuously differentiable and strictly increasing function defined on $[0, T]$, and $h_1(t) > 0$ is a time variable point delay. For convenience, consider the time leading function $r(t) = t + h(t)$, which is the inverse function for $v(t)$. That is, we have $r(v(t)) = t$. Note that for $t \in [0, v(T)]$ system (1) is in fact a system without delay. Hence, this point onwards we will assume that $v(T) > 0$.

We find many results on exact controllability of infinite dimensional systems, which are summarized by Balachandran and Dauer [1]. Controllability of infinite dimensional systems with time variable delays in control is studied by Klamka [2]. Klamka studied controllability of linear stochastic systems with fixed time delay in control in [3] and with time variable delay in control in [4]. Shen and Sun [5] studied relative controllability of nonlinear stochastic systems with delay in control. Controllability of deterministic semilinear systems with fixed delay in control is studied by Kumar and Sukavanam [6]. This article is a part of Ph. D. Thesis [12].

In this article we will study the controllability of stochastic nonlinear infinite dimensional systems with time variable delay in control. In section 2, some preliminaries are discussed. Controllability of stochastic linear infinite dimension systems with variable delay control are studied in section 3, and nonlinear systems in section 4. Finally in section 5, some examples are discussed.

II. Preliminaries

The mild solution of (1) is defined as

$$\begin{aligned} x(t) = \quad & S(t)x_0 + \int_0^t S(t-s)(B_1 u(s) + f(s, x(s), u(s)))ds \\ & + \int_0^t S(t-s)B_2 u(v(s))ds + \int_0^t S(t-s)g(s, x(s), u(s))dw(s). \end{aligned} \tag{3}$$

count the zero initial control for $t \in [v(0), 0]$, the mild solution (3) can be written as

$$x(t) = S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t-r(s))B_2 r'(s))u(s)ds + \int_{v(t)}^t S(t-s)B_1 u(s)ds + \int_0^t S(t-s)f(s,x(s),u(s))ds + \int_0^t S(t-s)g(s,x(s),u(s))dw(s). \quad (4)$$

Let $\mathcal{U}_{ad} = \mathcal{L}_2^{\mathcal{F}_t}([0,T]\times\Omega,\mathcal{U})$. Define the set of all reachable states as follows:

$$R_T(T;x_0,u) = \{x(T;x_0,u) | u(\cdot)\in\mathcal{U}_{ad},\ x_0\in\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H})\}.$$

Definition 1 *The controlled stochastic system (1) is said to be relatively exactly controllable on* $[0,T]$ *if for every initial condition* $x_0 \in \mathcal{L}_2(\Omega,\mathcal{F}_0,H)$, *there is some control* $u\in\mathcal{U}_{ad}$ *such that* $R(T;x_0,u) = \mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H})$.

Definition 2 *The controlled stochastic system (1) is said to be relatively approximately controllable on* $[0,T]$ *if for every initial condition* $x_0\in\mathcal{L}_2(\Omega,\mathcal{F}_0,H)$, *there is some control* $u \in \mathcal{U}_{ad}$ *such that* $\overline{R(T;x_0,u)} = \mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H})$.

It $T>0$ is arbitrarily small, then we say that system is small time relatively exactly controllable, and small time relatively approximately controllable.

Define the linear bounded control operator $L_T\in\mathcal{L}(\mathcal{U}_{ad},\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}))$ as follows:

$$L_T u = \int_0^{v(T)} (S(T-t)B_1 + S(T-r(s))B_2 r'(s))u(s)ds + \int_{v(T)}^T S(T-s)B_1 u(s)ds.$$

The adjoint $L_T^*: \mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}) \to \mathcal{U}_{ad}$ of L_T is

$$L_T^* = (B_1^* S^*(T-t) + B_2^* S^*(T-v(t))r'(t))\mathbb{E}\{\cdot\,|\mathcal{F}_t\}, \qquad t\in[0,v(T)],$$
$$L_T^* = B_1^* S^*(T-t)\mathbb{E}\{\cdot\,|\mathcal{F}_t\}, \qquad t\in(v(T),T].$$

We observe that

$$R(T;x_0,u) = S(T)x_0 + \mathrm{Im}L_T + \int_0^T S(T-s)f(s,x(s),u(s))ds + \int_0^T S(T-s)g(s,x(s),u(s))dw(s).$$

The deterministic controllability operator is defined as

$$\Psi_s^T = \int_s^{v(T)} (r'(t)S(T-r(t))B_2B_2^*S^*(T-r(t)r'(t) + S(T-t)B_1B_1^*S^*(T-t))dt + \int_{v(T)}^T S(T-t)B_1B_1^*S^*(T-t)dt, \qquad s<v(T),$$

$$\Psi_s^T = \int_s^T S(T-t)B_1B_1^*S^*(T-t)dt,\ s\ge v(T),$$

and linear controllability operator $\Pi_0^T \in \mathcal{L}(\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}),\mathcal{L}_2(\Omega,\mathcal{F}_T,\mathcal{H}))$ associated with (1) is defined as

$$\Pi_0^T = \int_0^{v(t)} (r'(t)S(T-r(t))B_2B_2^*S^*(T-r(t)r'(t) + S(T-t)B_1B_1^*S^*(T-t))\mathbb{E}\{\cdot\,|\mathcal{F}_t\}dt + \int_{v(t)}^{T} S(T-t)B_1B_1^*S^*(T-t)\mathbb{E}\{\cdot\,|\mathcal{F}_t\}dt.$$

We assume

Hypothesis 1

1. The functions $f: [0,T] \times \mathcal{H} \times \mathcal{U} \to \mathcal{H}$, $g: [0,t] \times \mathcal{H} \times \mathcal{U} \to \mathcal{L}_Q(\mathcal{K},\mathcal{H})$ satisfy the Lipschitz condition. That is, there exists some positive constant L such that for all $x_1,\ x_2 \in \mathcal{H},\ u_1,\ u_1 \in \mathcal{U},\ t \in [0,T]$
$$\| f(t,x_1,u_1) - f(t,x_2,u_2) \|^2 + \| g(t,x_1,u_1) - g(t,x_2,u_2) \|_Q^2 \leq L(\| x_1 - x_2 \|^2 + \| u_1 - u_2 \|^2).$$
2. The functions f, g are continuous on $[0,T] \times \mathcal{H} \times \mathcal{U}$, and there exists some positive constants $L > 0$ such that for all $x \in \mathcal{H},\ u \in \mathcal{U},\ t \in [0,T]$
$$\| f(t,x,u) \|^2 + \| g(t,x,u) \|_Q^2 \leq L(1 + \| x \|^2 + \| u \|^2).$$
3. f and g are bounded on $[0,T] \times \mathcal{H} \times \mathcal{U}$.

Under the hypothesis 1, for any $u \in \mathcal{U}_{ad}$ there existence the unique mild solution to an integral equation (4) [7].

III. Controllability of linear systems

In this section, we will consider the linearized system

$$dx(t) = Ax(t) + B_1u(t) + B_2u(v(t)) + g(t)dw(t), \qquad t \in [0,T], \tag{5}$$

with initial conditions

$$x(0) = x_0 \in \mathcal{L}_2(\Omega,\mathcal{F}_0,\mathcal{H}) \qquad \text{and} \qquad u(t) = 0, \qquad t \in [v(0),0]. \tag{6}$$

Also, consider the infinite dimensional deterministic system with time-variable delay in control given by

$$dy(t) = Ay(t) + B_1w(t) + B_2w(v(t)), \qquad t \in [0,T], \tag{7}$$

with initial conditions

$$y(0) = y_0 \in \mathcal{L}_2(\Omega,\mathcal{H}) \qquad \text{and} \qquad w(t) = 0, \qquad t \in [v(0),0], \tag{8}$$

where $w \in L_2([0,T],\mathcal{U})$ is admissible control.

Following theorem gives necessary and sufficient condition for the relative exact controllability of linear stochastic system (5).

Theorem 1 *The stochastic system (5) is relatively exactly controllable on* $[0,T]$ *if and only if any one of the following conditions holds.*

1. $\Pi_0^T \geq \gamma I$.
2. $R(\lambda, \Pi_0^T)$ converges as $\lambda \to 0^+$ in uniform operator topology.
3. $\lambda R(\lambda, \Pi_0^T)$ converges to the zero as $\lambda \to 0^+$ in uniform operator topology.

Proof. The proof is similar to the proof of Theorem 1 in [8].

Following theorem, similar to the Theorem 3.2 in [9] relates the relative exact controllability of deterministic system (7) and stochastic system (5).

Theorem 2 *The following conditions are equivalent:*

1. The stochastic system (5) relatively exactly controllable on $[0,T]$.
2. The deterministic system (7) is relatively exactly controllable on every $[s,T]$, $0 \leq s \leq T$.
3. The deterministic system (7) is small time relatively exactly controllable.
4. The stochastic system (5) is small time relatively exactly controllable.

Proof. (1)⇒(2). Assume that the stochastic system (5) is relatively exactly controllable on $[0,T]$. Then by Theorem 1,

$$\mathbb{E}\langle \Pi_0^T x, x\rangle \geq \gamma \mathbb{E} \parallel x|^2 \quad \text{forsome} \gamma > 0 \quad \text{andall} x \in \mathcal{H}.$$

Using Lemma 2.3 from [9], we have

$$\mathbb{E}\langle \Pi_0^T x, x\rangle = \mathbb{E}\left\langle \Psi_0^T \mathbb{E}x + \sum_{j=1}^{k} \int_0^T \Psi_s^T \phi_j(s) dw_j(s), \mathbb{E}x + \sum_{j=1}^{k} \int_0^T \phi_j(s) dw_j(s) \right\rangle$$

$$= \langle \Psi_0^T \mathbb{E}x, \mathbb{E}x\rangle + \mathbb{E}\sum_{j=1}^{k} \alpha_j \int_0^T \langle \Psi_s^T \phi_j(s), \phi_j(s)\rangle ds$$

$$\geq \gamma \left(\parallel \mathbb{E}x \parallel^2 + \mathbb{E}\sum_{j=1}^{k} \alpha_j \int_0^T \parallel \phi_j(s) \parallel^2 ds \right).$$

Now, if $\mathbb{E}x = 0$ and $\phi(s)$ is such that

$$\phi_1(s) = \begin{cases} h & \text{if} s \in [r, r+\epsilon) \\ 0 & \text{otherwise} \end{cases}$$

and $\phi_j(\tau) = 0$, $j = 2,3,\cdots,k$, $\tau \in [0,T]$, then

$$\int_r^{r+\epsilon} \langle \Psi_s^T \phi_1(s), \phi_1(s)\rangle ds \geq \int_r^{r+\epsilon} \parallel \phi_1(s) \parallel^2 ds.$$

Dividing through by ϵ, and taking the limit as $\epsilon \to 0^+$ we obtain

$$\langle \Psi_r^T h, h \rangle \geq \gamma \parallel h \parallel^2, \text{ forsome} \gamma > 0.$$

That is, the deterministic system (7) is relatively exactly controllable on each $[r, T]$.

(2)⇒(3). It is clear from definitions.

(3)⇒(4). Suppose the deterministic system (7) is small time relatively exactly controllable. Then the operator Ψ_s^r is invertible and an operator

$$\wedge_0^\tau x = (\Psi_0^r)^{-1} \mathbb{E}x + \int_0^\tau (\psi_s^r)^{-1} \phi(s) dw(s)$$

is the inverse of Π_0^τ. The invertability of Π_0^τ for all $\tau > 0$ implies small time controllability of the stochastic system (5) .

(4)⇒(1). It is clear from definitions.

Following theorem gives necessary and sufficient condition for the relative approximate controllability of linear stochastic system (5).

Theorem 3 *The stochastic system (5) is relatively approximately controllable on* $[0, T]$ *if and only if any one of the following conditions holds.*

1. $\Pi_0^T > 0$.
2. $\lambda R(\lambda, \Pi_0^T)$ converges to the zero as $\lambda \to 0^+$ in the strong operator topology.
3. $\lambda R(\lambda, \Pi_0^T)$ converges to the zero as $\lambda \to 0^+$ in the weak operator topology.

Proof. The proof is similar to the proof of Theorem 1 in [10].

Following theorem, similar to the Theorem 4.2 in [9] relates the relative approximate controllability of deterministic system (7) and stochastic system (5).

Theorem 4 *The following conditions are equivalent:*

1. The stochastic system (5) relatively approximately controllable on $[0, T]$.
2. The deterministic system (7) is relatively approximately controllable on every $[s, T]$, $0 \leq s \leq T$.
3. The deterministic system (7) is small time relatively approximately controllable.
4. The stochastic system (5) is small time relatively approximately controllable.

Proof. (1)⇒(2). Let the system (5) be relatively approximately controllable on $[0, T]$. Then by Theorem 3

$$\mathbb{E} \parallel \lambda R(\lambda, \Pi_0^T) x \parallel^2 \to 0 \quad \text{as} \lambda \to 0^+.$$

From Lemma 2.3 in [9], we have

$$\mathbb{E} \parallel \lambda R(\lambda, \Pi_0^T) x \parallel^2 = \parallel \lambda R(\lambda, \Psi_0^T) \mathbb{E} x \parallel^2 + \mathbb{E} \sum_{j=1}^{k} \alpha_j \int_0^T \parallel \lambda R(\lambda, \Psi_s^T) \phi_j(s) \parallel^2 ds \to 0. \tag{9}$$

So

$$\mathbb{E} \sum_{j=1}^{k} \alpha_j \int_0^T \parallel \lambda R(\lambda, \Psi_s^T) \phi_j(s) \parallel^2 ds \to 0$$

for all $\phi \in \mathcal{L}_2^{\mathcal{F}}(0, T; \mathcal{L}_2(\mathbb{R}^k, \mathcal{H}))$. Hence, there is a subsequence $\{\lambda_k\}$ such that for all $h \in \mathcal{H}$,

$$\parallel \lambda_k R(\lambda_k, \Psi_s^T) h \parallel \to 0, \text{ almosteverywhereon } [0, T].$$

As $R(\lambda, \Psi_s^T)$ is continuous, above property holds for all $s \in [0, T)$. Thus the deterministic system (7) is relatively approximately controllable on every $[s, T]$, $0 \le s < T$.

(2)⇒(3). It is clear from definitions.

(3)⇒(4). Suppose the deterministic system (7) is small time relatively approximately controllable on every $[s, \tau]$. Then

$$\parallel \lambda R(\lambda, \Psi_s^\tau) \parallel \to 0 \quad \text{as} \lambda \to 0^+.$$

But

$$\sum_{j=1}^{k} \alpha_j \parallel \lambda R(\lambda, \Psi_s^\tau) \phi_j(s) \parallel^2 \le \sum_{j=1}^{k} \alpha_j \parallel \phi_j(s) \parallel^2.$$

So by Lebesgue dominated convergence theorem and (9), we have

$$\mathbb{E} \parallel \lambda R(\lambda, \Pi_0^\tau) \parallel^2 \to 0 \quad \text{as} \lambda \to 0^+.$$

Thus, the stochastic system (5) is small time approximately controllable.

(4)⇒(1). It is clear from definitions.

IV. Controllability of nonlinear systems

We have the following representation theorem.

Lemma 1 ([9]) *For any $h \in \mathcal{L}_2(\mathcal{F}_T, \mathcal{H})$ there exists a unique $\phi \in \mathcal{L}_2^{\mathcal{F}}([0, T], \mathcal{L}_Q(\mathcal{K}, \mathcal{H}))$ such that*

$$h = \mathbb{E} h + \int_0^T \phi(s) dw(s). \tag{10}$$

Lemma 2 *For arbitrary $f(s) \in \mathcal{L}_2^{\mathcal{F}}([0, T], \mathcal{H})$, $g(s) \in \mathcal{L}_2^{\mathcal{F}}([0, T], \mathcal{L}_Q(\mathcal{K}, \mathcal{H}))$, $h \in \mathcal{L}_2(\mathcal{F}_T, \mathcal{H})$ the control*

$$u(t) = B_1^* S^*(T - t)(\alpha + \Psi_0^T)^{-1}(\mathbb{E} h - S(T) x_0) - B_1^* S^*(T - t) \int_0^{v(T)} (\alpha + \Psi_s^T)^{-1} S(T - s) f(s) ds - B_1^* S^*(T - t) \int_0^{v(T)} (\alpha + \Psi_s^T)^{-1} [S(T - s) g(s) - \phi(s)] dw(s), \; t \in [0, v(T)],$$

and

$$u(t) = (B_1^* S^*(T - t) + B_2^* S^*(T - r(t) r'(t))(\alpha + \Psi_0^T)^{-1}(\mathbb{E} h - S(T) x_0) - (B_1^* S^*(T - t) + B_2^* S^*(T - r(t) r'(t)) \int_{v(T)}^T (\alpha + \Psi_s^T)^{-1} S(T - r(s)) f(s) ds - (B_1^* S^*(T - t) + B_2^* S^*\left(T - r(t) r'(t)\right) \times$$

$$\int_{v(T)}^{T} (\alpha + \Psi_s^T)^{-1}[S(T - r(s))g(s) - \phi(s)]dw(s),\ t \in [v(T), T], \tag{11}$$

transfers the system

$$\begin{aligned} x(t) = \quad & S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t - r(s))B_2 r'(s))u(s)ds \\ & + \int_{v(t)}^{t} S(t-s)B_1 u(s)ds + \int_0^t S(t-s)f(s)ds + \int_0^t S(t-s)g(s)dw(s) \end{aligned} \tag{12}$$

from $x_0 \in \mathcal{H}$ to

$$\begin{aligned} x(T) = \quad & h - \alpha(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) - \alpha \int_0^T (\alpha + \Psi_s^T)^{-1} S(t-s)f(s)ds \\ & -\alpha \int_0^T (\alpha + \Psi_s^T)^{-1}[S(t-s)g(s) - \phi(s)]dw(s) \end{aligned} \tag{13}$$

at time T. Here ϕ comes from Lemma 1.

Proof. By substituting (11) and into (12) and using Fubini theorem we get (13) (see Lemma 4 in [11]).

Define the operator $\Phi_\alpha : \mathcal{H} \times \mathcal{U}_{ad} \to \mathcal{H} \times \mathcal{U}_{ad}$ as follows:

$$(z^\alpha(t), \beta^\alpha(t)) = \Phi_\alpha(x, u)(t), \tag{14}$$

where

$$\begin{aligned} z^\alpha(t) = \quad & S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t - r(r))B_2 r'(s))\beta^\alpha(s)ds + \int_{v(t)}^{t} S(t-s)B_1 \beta^\alpha(s)ds \\ & + \int_0^t S(t-s)f(s, x(s), u(s))ds + \int_0^t S(t-s)g(s, x(s), u(s))dw(s), \end{aligned}$$

where

$$\begin{aligned} \beta^\alpha = {} & B_1^* S^*(T-t)(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) \\ & -B_1^* S^*(T-t) \int_0^{v(T)} (\alpha + \Psi_s^T)^{-1} S(T - s)f(s, x(s), u(s))ds \\ & -B_1^* S^*(T-t) \int_0^{v(T)} (\alpha + \Psi_s^T)^{-1}[S(T - s)g(s, x(s), u(s)) - \phi(s)]dw(s),\ t \in [0, v(T)], \end{aligned}$$

and

$$\begin{aligned} \beta^\alpha = {} & (B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t))(\alpha + \Psi_0^T)^{-1}(\mathbb{E}h - S(T)x_0) \\ & - (B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t)) \int_{v(T)}^{T} (\alpha + \Psi_s^T)^{-1} S(T - r(s))f(s, x(s), u(s))ds \\ & -(B_1^* S^*(T-t) + B_2^* S^*(T - r(t)r'(t)) \\ & \times \int_{v(T)}^{T} (\alpha + \Psi_s^T)^{-1}[S(T - r(s))g(s, x(s), u(s)) - \phi(s)]dw(s),\ t \in [v(T), T], \end{aligned} \tag{15}$$

Existence of unique fixed point to the operator is proved in the following theorem.

Theorem 5 *Assume that Hypothesis 1 is true. Then for any* $\alpha > 0$ *the operator* Φ_α *has a unique* *fixed point.*

Proof. Proof is similar to the proof of Theorem 6 in [11].

Now, we prove relative approximate controllability of nonlinear stochastic system (1).

Theorem 6 *Assume that Hypothesis 1 is true and linear stochastic system (5) is relatively approximately controllable, then nonlinear stochastic system (1) is relatively approximately controllable.*

Proof. Proof is similar to the the proof of Theorem 7 in [11].

Corollary 1 *Assume that Hypothesis 1 is true. If the semigroup S(t) is analytic and the deterministic linear system (7) is relatively approximately controllable on* $[0,T]$ *then nonlinear stochastic system (1) is relatively approximately controllable on* $[0,T]$.

Proof. It is well known that, the semigroup $S(t)$ is analytic and the linear stochastic system (5) is relatively approximately controllable on $[0,T]$ if and only if the deterministic linear system (7) relatively approximately controllable on $[0,T]$ (See [9], Theorem 4.3). Then by Theorem 6 nonlinear stochastic system is relatively approximately controllable.

Define the operator $\Phi^0 : \mathcal{H} \times \mathcal{U}_{ad} \to \mathcal{H} \times \mathcal{U}_{ad}$ as follows:

$$(z(t), \beta(t)) = \Phi^0(x,u)(t)$$

where

$$z(t) = S(t)x_0 + \int_0^{v(t)} (S(t-s)B_1 + S(t-r(r))B_2 r'(s))\beta(s)ds + \int_{v(t)}^t S(t-s)B_1\beta(s)ds + \int_0^t S(t-s)f(s,x(s),u(s))ds + \int_0^t S(t-s)g(s,x(s),u(s))dw(s),$$

where

$$u(t) = B_1^* S^*(T-t)\mathbb{E}\{(\Pi_0^T)^{-1}p(x) | \mathcal{F}_t\},\ t \in [0, v(T)],$$

where

$$p(x) = x_T - S(T)x_0 - \int_0^{v(T)} S(T-s)f(s,x(s),u(s))ds - \int_0^{v(T)} S(T-s)g(s,x(s),u(s))dw(s),$$

and

$$u(t) = (B_1^* S^*(T-t) + B_2^* S^*(T-r(t)r'(t))\mathbb{E}\{(\Pi_0^T)^{-1}p(x) | \mathcal{F}_t\}, \quad t \in [v(T),T],$$

where

$$p(x) = x_T - S(T)x_0 - \int_{v(T)}^T S(T-r(s))f(s,x(s),u(s))ds - \int_{v(T)}^T S(T-r(s))g(s,x(s),u(s))dw(s).$$

Now, we are ready to prove relative exact controllability of (1).

Theorem 7 *Assume that Hypothesis 1 holds and the linear stochastic system (5) is relatively exactly controllable. Then the operator* Φ^0 *has a unique fixed point.*

Proof. Proof is similar to the proof of Theorem 5.

Theorem 8 *Assume that Hypothesis 1 is true and the linear stochastic system (5) is relatively exactly controllable on* $[0,T]$, *then the nonlinear stochastic system (1) is relatively exactly controllable.*

Proof. By Theorem 7, there exists a unique fixed point of an operator Φ^0. Let $(x^0,u^0)(\cdot)$ be the unique fixed point of an operator Φ^0. Then $x_T^0 = x_T$ for arbitrary $x_T \in \mathcal{L}_2(\mathcal{F}_T,\mathcal{H})$. Thus system (1) is relatively exactly controllable on $[0,T]$.

Corollary 2 *Assume that Hypothesis 1 is true. If the deterministic linear system (7) is relatively exactly controllable on all* $[0,t]$, $t>0$, *then nonlinear stochastic system (1) is relatively exactly controllable on* $[0,T]$.

Proof. By Theorem 1, the linear stochastic system (5) is relatively exactly controllable on $[0,T]$ if and only if the deterministic linear system (7) small time relatively exactly controllable on $[0,T]$, that is, relatively exactly controllable on all $[0,t]$, $t>0$. Then by Theorem 7 nonlinear stochastic system is relatively exactly controllable on $[0,T]$.

V. Examples

Example 1 *Consider the following parabolic stochastic partial differential equation:*

$$\begin{aligned} dy(t,\xi) = &\ [y_{\xi\xi}(t,\xi) + B_1u(t,\xi) + B_2u(v(t),\xi) + f(t,y(t,\xi),u(t,\xi))]dt \\ &+g(t,y(t,\xi),u(t,\xi))dw(s),\ t\in[0,T],\ \xi\in[0,\pi], \\ y(t,0) = &\ y(t,\pi)=0,\ t>0. \end{aligned} \tag{16}$$

Let $\mathcal{H}=\mathcal{K}=L_2([0,\pi])$. Define $Ay=y''$ with

$$D(A)=\{y\in\mathcal{H}|\quad y,\ y_\xi \quad \text{areabsolutelycontinuous}, y_{\xi\xi}\in\mathcal{H},\ y(0)=0,\ y(\pi)=0\},$$

then

$$Ay=\sum_{n=1}^{\infty}(-n^2)(y,\ e_n(\eta))e_n(\eta),\ y\in D(A)$$

with $e_n(\eta)=\sqrt{2/\pi}\sin n\eta$, $n=1,\ 2,\ 3,\ \cdots$, $e_0=1$.

It is well know that A generates a strongly continuous semigroup $S(t)$, $t>0$.

Define an infinite dimensional space

$$\mathcal{U}=\{u=\sum_{n=2}^{\infty}u_ne_n \quad \text{with} \sum_{n=2}^{\infty}u_n^2<\infty\}$$

with norm $\| u \| = (\sum_{n=2}^{\infty}u_n^2)^{\frac{1}{2}}$. Define a linear continuous mapping B_1 from $\mathcal{U}$ to $\mathcal{H}$ as follows

$$B_1u=2u_2e_1(\eta)+\sum_{n=2}^{\infty}u_ne_n(\eta).$$

The system (16) can be written in abstract form given by (1) with $B_2=I$. Then relative approximate controllability of linear system associated with (16) follows from Theorem 2. In addition, if Hypothesis 1 is true. Then relative approximate controllability of (16) follows from Theorem 6.

Example 2 *Consider the following hyperbolic stochastic partial differential equation:*

$$\begin{aligned} d(\partial/\partial t)y(t,\xi) = &\quad [y_{\xi\xi}(t,\xi) + B_1u(t,\xi) + B_2u(v(t),\xi) + f(t,y(t,\xi),u(t,\xi))]dt \\ &\quad +g(t,y(t,\xi),u(t,\xi))dw(s),\ t \in [0,T],\ \xi \in [0,\pi], \\ y(t,0) = &\quad y(t,1) = 0,\ t > 0, \\ y(0,\xi) = &\quad \mu(\xi),\ (\partial/\partial t)y(0,\xi) = \nu(\xi). \end{aligned} \tag{17}$$

Let $\mathcal{H} = D(A^{1/2}) \oplus L_2([0,1])$, endowed with the inner product

$$\langle w,v\rangle = \left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = \sum_{n=1}^{\infty} \{n^2\pi^2\langle w_1,e_n\rangle\langle e_n,v_1\rangle + \langle w_2,e_n\rangle\langle e_n,v_2\rangle\}.$$

where $e_n(\eta) = \sqrt{2}\sin n\pi\eta,\ n = 1,\ 2,\ 3,\ \cdots$. Let

$$z = \begin{bmatrix} y \\ (\partial/\partial t)y \end{bmatrix},\ z(0) = \begin{bmatrix} \mu \\ \nu \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix},\ G = \begin{bmatrix} 0 \\ g \end{bmatrix},\ F = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Define $A_0y = (d^2/d\xi^2)y$ and

$$A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix},$$

with

$D(A_0) = \{y \in L_2([0,1])|\ y, y_\xi$ are absolutely continuous, $y_{\xi\xi} \in \mathcal{H}, y(0) = 0, y(1) = 0\}$.

then system (17) can be written as

$$dz = (Az + B_1u + B_2u(v) + F(t,z,u))dt + G(t,z,u)dw,\ z(0) = \begin{bmatrix} \mu \\ \nu \end{bmatrix}. \tag{18}$$

It is well know that A is the infinitesimal generator of a contraction semigroup $S(t),\ t > 0$.

It is well known that the linear stochastic system associated with (18) is relatively exactly controllable [9]. In addition, if Hypothesis 1 is true. Then relative exact controllability (18) and hence that of (17) follows from Theorem 8.

References

K. Balachandran and J.P. Dauer, Controllability of nonlinear systems in Banach Spaces: a survey, J. Optim. Theory Appl., Vol. 115(2002), 7–28.

J. Klamka, Controllability of infinite dimensional systems with delays in control, Annal. Math. Sil., Vol. 1(1985), 51–65.

J. Klamka, Stochastic controllability and minimum energy control of systems with multiple delays in control, Appl. Math. Comp., Vol. 206(2008), 704–715.

J. Klamka, Stochastic controllability of systems with variable delay in control, Bull. Pol. Aca. Sci.: Tech. Sci., Vol. 56(2008), 279–284.

L. Shen and J. Sun, Relative controllability of stochastic nonlinear systems with delay in control, Nonlinear Anal., Vol. 13(2013), 2880–2887.

S. Kumar and N. Sukavanam, Controllability of semilinear systems with fixed delay in control, Opuscula Math., Vol. 35(2015), 71–83.

G. D. Prato and J. Zabczyk, Stochastic equations in infinite dimensions, Cambridge Uni. Press(1992).

N. I. Mahmudov and A. Denker, On controllability of linear stochastic systems, Int. J. Control, Vol. 73(2000), 144–151.

N. I. Mahmudov, Controllability of linear stochastic systems in Hilbert spaces, J. Math. Anal. Appl., Vol. 259(2001), 64–82.

A. Lindquist, On feedback Control of linear stochastic systems, SIAM J. Control, Vol. 11(1973), 323–343.

N. I. Mahmudov, Controllability of semilinear stochastic systems in Hilbert spaces, J. Math. Anal. Appl.l, Vol. 288(2003), 197–211.

R. B. Sonawane, Controllability and observability of stochastic partial differential equations, SRTM University Nanded(2015).

Cite Your Article as:

R. B. Sonawane. (2023). STOCHASTIC CONTROLLABILITY OF NONLINEAR SYSTEMS WITH TIME VARIABLE DELAY IN CONTROL. Scholarly Research Journal for Interdisciplinary Studies, 11(77), 427–438. https://doi.org/10.5281/zenodo.8214247